On optimization of long-term irreversible investments in a diffusion model

Elena V. Boguslavskaya

In [6] R. Pindyck introduced a model where uncertainty arise from the unknown amount of investments needed to complete a project. In this paper, we obtain an explicit solution for this problem.

To find a solution we use heuristic arguments based on the Bellman equation and the "smooth pasting condition". To prove optimality of the solution we use verification theorems of stochastic optimal control.

Key words and phrases: optimal control of investments, Bellman equation, smooth pasting conditions, Bessel functions, confluent hypergeometric functions, hypergeometric functions.

1 Introduction. Posing the problem

We consider the following mathematical model which was proposed by Pindyck in [6]. Let X_t be the cost remaining at time t to complete the project. We split the uncertainty about remaining investment into two components: *a technical uncertainty* that depends only on the firm's strategy and *an input cost uncertainty* that depends on external circumstances.

Let us assume that $X = (X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = -I_t dt + \beta\sqrt{X_t I_t} dW_t + \gamma X_t d\tilde{W}_t, \tag{1}$$

where I_t is a nonanticipating function (investment rate), β, γ are certain nonnegative constants, W_t and $\tilde{W}_t$ are uncorrelated Wiener processes.

Equation (1) shows that the cost needed to complete the project declines as investments proceed and simulataneously changes affected by two different types of uncertainty.

A. The case $\beta = 0, \gamma \neq 0$ corresponds to the input cost uncertainty. Here X can fluctuate even when there is no investment. The mathematical expectation of the variation $\mathsf{E}\left((dX)^2 \mid X\right) = \gamma^2 X^2 dt$ does not depend on investment rate I. For example, fluctuations in cost of labor and material, goverment regulations can influence X irrespectively of what the firm does.

B. The case $\beta \neq 0, \gamma = 0$ corresponds to technical uncertainty. Here X can fluctuate only if investments are taking place. The mathematical expectation of the variation $\mathsf{E}\left((dX)^2 \mid X\right) = \beta^2 I X dt$ is linear in I.

C. The case $\beta \neq 0, \gamma \neq 0$ allows for both types of uncetainty.

In all three cases the total amount of investment is known only after completion of the project. The parameters β, γ are chosen in accordance with the given applied problem (see [6]).

The rate of investment is a control in our problem. Our purpose is to find an optimal investment strategy for the project. The rate of investment I_t is a nonanticipating random function, which is nonnegative and bounded: $0 \leq I_t \leq I_{max}$, where I_{max} is a constant. A control that satisfies these conditions is called *an admissible control.* Let us denote the class of admissible controls by $\mathbf{U}$. Nonnegativity of the investment rate means that we cannot take back any part of our money once we have invested. The maximal value of the investment rate I_{max} is specified by two factors. The first one is that an investor can put his money only with a bounded rate. For example, a company cannot invest in a project more than 1$ a year. Besides that there can be external limitations on the investment rate. For example, construction works cannot be completed within certain time limit or research on a new drug cannot be speeded up by additional investments.

Let $F(x, I)$ denote *a utility functional*:

$$F(x, I) = \mathsf{E}_x\{\int_0^\tau (-I)e^{-rt}dt + Ve^{-r\tau}\}, \tag{2}$$

where E_x is the mathematical expectations w.r.t. $X_0 = x$, $\tau = \inf\{t : X_t = 0\}$,

$V = const > 0$ - *the value of the project upon completion*, which is known in advance,

$r = const \geq 0$ - investment rate.

The utility function is the way to measure the mathematical expectation of "gain from the project minus the investments". The time τ is *the time of finishing the project* ($X_\tau = 0$). The integral in (2) denotes the discounted investment in the project, and the second summand is the discounted gain from the project.

Let us introduce *a profit function*

$$F(x) = \sup_{I \in \mathbf{U}} F(x, I), \tag{3}$$

We consider $F(x)$ as a criterium for investment optimality. Note, that if the set of admissible control values was unbounded (i.e. $I_{max} = \infty$), then the optimal strategy is to invest immediately all the capital needed to complete the project.

Further note that $F(x, I, V) = I_{max}F(x/I_{max}, I/I_{max}, V/I_{max})$. Thus we can put $I_{max} = 1$ without loss of generality.

In this paper we want to find an optimal strategy of investment $\tilde{I} = \tilde{I}(x)$ and the profit function $F = F(x)$. Generally, problems of this type (see [8], [5]) are solved in the following way. Using heuristic arguments one finds a strategy which is suspected to be optimal. Then one computes the corresponding value of the utility functional. Finally, one proves that the strategy is optimal and that the function constructed is indeed the profit function. We prove that the following strategy is optimal: *if the capital needed to complete the project is less than a certain value x^*, then we should invest at the maximal rate; if at any moment the capital needed to complete the project exceeds x^*, we should stop investing in a project.*

Pindyck [6] has found the solution for the case of technical uncerainty when the interest rate is equal to zero ($r = 0$). We obtain an explicit solution for the general problem when $r > 0$. It is interesting to note that in the case of input cost uncertainty and in the presence of two uncertainties the optimal strategy will be *to not invest at all*, but to wait while the project is completed "by itself"! Indeed, here the time is "free" ($r = 0$) and the probability that the process (1) with $I_t \equiv 0$ hits zero in finite time (the probability of completing the project in finite time) is equal to 1.

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2 Finding the solution

2.1 Formulating Stefan problem with a free boundary

The value function $F = F(x)$ is the criterium of optimality in our model:

$$F(x) = \sup_{I \in \mathbf{U}} \mathsf{E}_x\{\int_0^\tau (-I)e^{-rt}dt + Ve^{-r\tau}\}. \tag{4}$$

Here $\mathbf{U} = \{I : 0 \leq I_t \leq 1\}$, $\tau = \inf\{t : X_t = 0\}$. One can easily see from (4) that $F(0) = V$ and $F(x) \leq V$ for all $x \geq 0$.

Let us assume that there exists a function $\tilde{F} = \tilde{F}(x)$ and a control $\tilde{I} = \tilde{I}(x)$ such that the Bellman equation is satisfied.

Let us introduce operators L_1 and L_2 acting on functions $G = G(x), x \geq 0$, from $(C^2(0,\infty)\backslash\{x^*\})\bigcup C^1(0,\infty)$, (where $x^* \in (0,\infty)$) according to the formula

$$L_1G = -\frac{dG}{dx} + \frac{1}{2}\beta^2 x\frac{d^2G}{dx^2} \tag{5}$$
$$L_2G = \frac{1}{2}\gamma^2 x^2\frac{d^2G}{dx^2} - rG$$

Let us write the Bellman equation

$$\sup_{0\leq I\leq 1}\{IL_1\tilde{F}(x) + L_2\tilde{F}(x) - I\} = 0. \tag{6}$$

As one can see equation (6) is linear in the control I. Therefore we may assume that the optimal control is the following:

$$\tilde{I}(x) \quad = \quad \begin{cases} 1, & L_1\tilde{F}(x) - 1 \geq 0 \\ 0, & L_1\tilde{F}(x) - 1 < 0 \end{cases}. \tag{7}$$

In this way we obtain that for those x, where $\tilde{I}(x) = 1$, the following equation is satisfied

$$L_1\tilde{F}(x) + L_2\tilde{F}(x) - 1 = 0, \tag{8}$$

and for those x, where $\tilde{I}(x) = 0$, the following equation holds

$$L_2\tilde{F}(x) = 0. \tag{9}$$

Taking into account intuitive considerations on the structure of the optimal control, we assume that there exists a constant x^*, such that $\tilde{I}(x) = 1$ for $x < x^*$, and $\tilde{I}(x) = 0$ for $x \geq x^*$ (i.e. we must invest at maximal rate if the cost of the project is "reasonable", and abandon the project when it becomes too "expensive"). In other words we have to solve the following free boundary Stefan problem: find a number x^* and a smooth function $\tilde{F}(x)$ such that

$$\tilde{F}(0) = V, \tag{10}$$
$$\tilde{F}(x) \leq V \text{ for all } x \geq 0, \tag{11}$$
$$L_1\tilde{F}(x) + L_2\tilde{F}(x) - 1 = 0, x \leq x^*, \tag{12}$$
$$L_2\tilde{F}(x) = 0, x \geq x^*. \tag{13}$$

2.2 Solution to the Stefan problem

One can easily check that the following conditions are sufficient and necessary for $\tilde{F}(x)$, $x \geq x^*$ to be a solution for the Stefan problem for $x \geq x^*$

$$\tilde{F}(x) = \frac{x}{(b-1)(\frac{1}{2}\beta^2 b + 1)} \left(\frac{x}{x^*}\right)^{-b} \quad x \geq x^*, \; b = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8r}{\gamma^2}}\right). \tag{14}$$

Indeed, the general solution of (13) is $c_1 x^{1-b} + c_2 x^b$. It follows from (11), that $c_2 = 0$. We can obtain c_1 from (12) with $x = x^*$. Note that the case $\gamma = 0$ may be considered as an asymptotic by letting $\gamma \to 0$.

Consider the following differential equation:

$$L_1 u(x) + L_2 u(x) = 0. \tag{15}$$

Suppose $u_1(x)$ and $u_2(x)$ are linearly independent solutions of (15), such that $u_1(0) = 1$, $u_2(0) = 0$. The following condition is necessary and sufficient for $\tilde{F}(x)$ with $0 \leq x \leq x^*$ to be a solution to the Stefan problem on $[0, x^*]$

$$\tilde{F}(x) = \left(V + \frac{1}{r}\right)[u_1(x) - \Theta(x^*)u_2(x)] - \frac{1}{r} \text{ when } 0 \leq x \leq x^*, \tag{16}$$

where $\Theta(x)$ and x^* are obtained from the following conditions

$$\tilde{F}'(x)\,|_{x=x^*} = -\frac{1}{\frac{1}{2}\beta^2 b + 1} \text{ (the "smooth pasting" condition)}, \tag{17}$$

$$L_2 \tilde{F}(x)\,|_{x=x^*} = 0. \tag{18}$$

Substituting (16) in (17) and (18) we get

$$\Theta(x) = \frac{L_2 u_1(x) - \left(\frac{1}{2}\beta^2 b + 1\right) u_1'}{L_2 u_2(x) - \left(\frac{1}{2}\beta^2 b + 1\right) u_2'}, \tag{19}$$

Note that using (15) and (5), one can write

$$\begin{aligned} L_2 u_i(x) - \left(\frac{1}{2}\beta^2 b + 1\right) u_i' &= -L_1 u_i(x) - \left(\frac{1}{2}\beta^2 b + 1\right) u_i' \\ &= -\frac{1}{2}\beta^2 \left(x u_i''(x) + b u_i'(x)\right), \text{ for } i = 1, 2. \end{aligned}$$

Thus, we can rewrite (19) as

$$\Theta(x) = \frac{x u_1''(x) + b u_1'(x)}{x u_2''(x) + b u_2'(x)}. \tag{20}$$

We shall obtain x^* as the minimal positive root of the equation $\Phi(x) = 0$, where

$$\Phi(x) = \left(V + \frac{1}{r}\right)[L_1 u_1(x) - \Theta(x) L_1 u_2(x)] - 1 \tag{21}$$

We shall show further that x^* is also the minimal positive root of the equation

$$L_1\tilde{F}(x) - 1 = 0 \text{ (see lemma 3.2)}$$

.

To find an explicit solution in terms of special functions we have to consider the cases of technical uncertainty, input cost uncertainty, and the case of the presence of two uncertainties separately. As equation (15) is an equation of type 2.1.2.166 in [7], we can write down its explicit solutions $u_1(x)$ and $u_2(x)$. To find an explicit expression for $\Theta(x)$ and $\Phi(x)$, we shall use identities from Appendices A, B, C for eq.(20) (21) in the cases of technical unceratinty, input cost unceratinty and in the case of the presence of both uncertainties respectively.

The case of technical uncertainty Let $c = 1 + 2/\beta^2$. Replace z by $z = 2rx/\beta^2$. We find

$$u_1(z) = \frac{2}{\Gamma(c)} z^{c/2} K_c(2\sqrt{z}), \quad (22)$$

$$u_2(z) = \frac{2}{\Gamma(c)} z^{c/2} I_c(2\sqrt{z}), \quad (23)$$

$$\Theta(z) = \frac{u_1'(z)}{u_2'(z)} \overset{(41),(42)}{=} -\frac{K_{c-1}(2\sqrt{z})}{I_{c-1}(2\sqrt{z})}, \quad (24)$$

where $I_\nu(x), K_\nu(x)$ are modified Bessel functions of the first and the second type respectivetly (see Appendix A). Note that $L_1 u_i - r u_i = 0, i = 1, 2$, since $\gamma = 0$. Thus we can write down the expression for $\Phi(z)$

$$\Phi(z) = (Vr + 1)\frac{2}{\Gamma(c)} z^{c/2} \left(K_c(2\sqrt{z}) + \frac{K_{c-1}(2\sqrt{z})}{I_{c-1}(2\sqrt{z})} I_c(2\sqrt{z}) \right) - 1. \quad (25)$$

The case of input cost uncertainty Let $z = \frac{r}{b(b-1)}x$. Then

$$u_1(z) = \frac{\Gamma(b+1)}{\Gamma(2b)} z^{1-b} e^{-1/z} M(b+1, 2b, z^{-1}), \quad (26)$$

$$u_2(z) = \frac{\Gamma(b+1)}{\Gamma(2b)} z^{1-b} e^{-1/z} U(b+1, 2b, z^{-1}) \quad (27)$$

where $M(a, b, x)$ and $U(a, b, x)$ are confluent hypergeometic functions of the first and the second type (see Appendix B). Use (50) and (51) to compute the first and the second derivatives from $u_1(z)$ and $u_2(z)$. Substitute the expressions for the derivatives in (20). Then (52) and (53) give

$$\Theta(z) = -\frac{b-1}{2} \frac{M(b, 2b+1, z^{-1})}{U(b, 2b+1, z^{-1})}. \quad (28)$$

Note that $L_1 u_i = -\frac{r}{b(b-1)} \frac{du_i}{dz}, i = 1, 2$ as $\beta = 0$. Take derivatives using (50) and (51) to obtain

$$\Phi(z) = (Vr + 1)\frac{\Gamma(b)}{\Gamma(2b)} z^{-b} e^{-1/z} \left(M(b, 2b, z^{-1}) - \frac{1}{2} \frac{M(b, 2b+1, z^{-1})}{U(b, 2b+1, z^{-1})} U(b, 2b, z^{-1}) \right) - 1. \quad (29)$$

The case of two uncertainties Let $z = \frac{c-1}{b(b-1)} rx$. We have

$$u_1(z) = F(b-1, -b; 1-c; -z) \tag{30}$$
$$u_2(z) = z^c F(b-1+c, -b+c; c+1; -z) \tag{31}$$

where $F(a, b; c, x)$ are hypergeometric functions (see Appendix C). Using (58) and (59) we compute the first and the second derivatives of $u_1(z)$ and $u_2(z)$. Then we substitute the expressions for the derivatives in (20). Thus from (62) and (63) we obtain

$$\Theta(z) = -\frac{(b-1)b^2}{c(c-1)(b+c-1)} z^{1-c} \frac{F(1+b, 1-b; 2-c; -z)}{F(b+c, -b+c; c; -z)}. \tag{32}$$

Using (64) and the expression for the first and the second derivatives of $u_i(z), i = 1, 2$, it is not difficult to obtain expressions for $L_1 u_i(z), i = 1, 2$. Substituting the obtained expressions in (21) we get

$$\begin{aligned} \Phi(z) &= (Vr+1)[F(b, 1-b; 1-c; -z) + \\ &+ \frac{b(b-c)}{c(c-1)} z \frac{F(1+b, 1-b; 2-c; -z)}{F(b+c, -b+c; c; -z)} F(b+c, 1-b+c; c+1; -z)] - 1. \end{aligned} \tag{33}$$

Now we can summarize the results in the following theorem:

Theorem 2.1 *In the model described above the optimal control $\tilde{I} = \tilde{I}(x)$ and the value function $F = F(x)$ are the following:*

$$\tilde{I}_t = \begin{cases} 1, & x < x^* \\ 0, & x \geq x^* \end{cases}$$

$$F(x) = \begin{cases} \left(V + \frac{1}{r}\right) [u_1(x) - \theta(x^*) u_2(x)] - 1/r, & x < x^* \\ \frac{x}{(b-1)(\frac{1}{2}\beta^2 b + 1)} \left(\frac{x}{x^*}\right)^{-b}, & x \geq x^*, \end{cases}$$

where $b = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8r}{\gamma^2}}\right)$, x^* is the minimal positive root of the equation $\Phi(x) = 0$, and $u_1(x), u_2(x), \Theta(x)$ and $\Phi(x)$ are defined by the equalities (22) —(33).

3 The proofs

In this section we prove theorem 2.1. We shall need two lemmas.

Lemma 3.1 *There exists at least one positive root of the equation $\Phi(z) = 0$.*[1]

Proof of the lemma. Note, that for any $z \geq 0, b > 1, c > 1$ we have

$$\begin{aligned} I_{c-1}(2\sqrt{z}) &> 0, \\ U(b, 2b+1, z^{-1}) &\overset{(integral\ transform)}{=} \frac{1}{\Gamma(b)} \int_0^\infty e^{-t/z} t^{b-1} (1+t)^b dt > 0, \\ F(b+c, -b+c; c; -z) &\overset{(Euler\ transform)}{=} (1+z)^{-b-c} F(b+c, b; c; \frac{z}{z+1}) > 0. \end{aligned}$$

[1]Note that the changes of variables $z = z(x)$ depend on the cases of uncertainty.

Therefore, $\Phi = \Phi(z)$ is a continuous function, the Bessel, confluent hypergeometric and hypergeometric functions are continuous. Taking the asymptotic representation of corresponding special functions we get that $\Phi(0) = Vr > 0$ and $\Phi(+\infty) = -1 < 0$. By the theorem from analysis there is a value z^* where $\Phi(z^*) = 0$. △

Lemma 3.2 *The following inequalities hold*

$$L_1\tilde{F}(x) - 1 > 0, x < x^*, \tag{34}$$

$$L_1\tilde{F}(x) - 1 \le 0, x \ge x^*. \tag{35}$$

PROOF OF THE LEMMA. Let us prove (35). For $\gamma = 0$ we have $L_1\tilde{F} - 1 = -1 < 0$. If $\gamma \neq 0$ we have

$$L_1\tilde{F}(x) - 1 = \left(\frac{x^*}{x}\right)^b - 1 \le 0, x \ge x^*.$$

Now let us prove (34) separately for each type of unceratainty.

Let $\Psi(z) \stackrel{def}{=} L_1\tilde{F}(x) - 1$. Then in the case of **technical uncertainty** we have

$$\Psi(z) = (Vr + 1)\frac{2}{\Gamma(c)}\left(z^{c/2}K_c(2\sqrt{z}) - \Theta(z^*)I_c(2\sqrt{z})\right) - 1.$$

Take the derivative of $\Psi(z)$. Note that $I_{c-1}(2\sqrt{z}) > 0$, for $c > 1$. By corrolary 7.1 from Appendix D we obtain the inequality

$$\begin{aligned}\Psi'_z(z) &= (Vr+1)\frac{2}{\Gamma(c)}\left[-z^{(c-1)/2}K_{c-1}(2\sqrt{z}) + \Theta(z^*)z^{(c-1)/2}I_{c-1}(2\sqrt{z})\right] = \\ &= (Vr+1)\frac{2}{\Gamma(c)}z^{(c-1)/2}I_{c-1}(2\sqrt{z})\left[\Theta(z) - \Theta(z^*)\right] \stackrel{(cor.7.1)}{<} 0.\end{aligned}$$

Thus, $\Psi(z)$ is a strictly decreasing continuous function for $z < z^*$. Moreover, by asymptotic properties of modified Bessel functions we have $\Psi(0) = Vr$. Also from the conditions of the lemma we have $\Psi(z^*) = 0$. Thus, $\Psi(z) > 0$ for $z < z^*$. △

In the case of **input cost uncertainty** we have:

$$\Psi(z) = (Vr+1)\frac{\Gamma(b)}{\Gamma(2b)}z^{-b}e^{-1/z}\left(M(b, 2b, z^{-1}) + \frac{\Theta(z^*)}{b-1}U(b, 2b+1, z^{-1})\right) - 1.$$

Let us consider the difference $\Psi(z) - \Phi(z)$. By corrolary 7.2 from Appendix D we have

$$\Psi(z) - \Phi(z) = (Vr+1)\frac{\Gamma(b-1)}{\Gamma(2b)}z^{-b}e^{-1/z}U(b, 2b; z^{-1})\left(\Theta(z^*) - \Theta(z)\right) > 0, z < z^*.$$

But for $z < z^*$ we have $\Phi(z) > 0$, as $\Phi(0) = Vr > 0$ and z^* is the minimal positive root of $\Phi(z) = 0$. Therefore, $\Psi(z) > \Phi(z) > 0$. △

In the case of **two uncertainties** we have:

$$\Psi(z)=(Vr+1)\left(F(b,1-b;1-c;-z)-\frac{(b-c)(b+c+1)}{b(b-1)}\Theta(z^*)z^cF(b+c,1-b+c;c+1;-z)\right)-1$$

In order to prove that $\Psi(z)$ is positive for $z<z^*$ we consider separately the cases $b-c<0$ and $b-c>0$.

Suppose $b-c<0$. Note, that

$$F(b+c,1-b+c;c+1;-z)\overset{Euler\ \ transform}{=}(1+z)^{-b-c}F(b+c,b;c+1;\frac{z}{1+z})>0$$

By corrolary 7.3 from Appendix D we obtain:

$$\Psi(z)-\Phi(z)=(Vr+1)\frac{b(b-c+1)}{b(b-1)}z^cF(b+c,1-b+c;c+1;-z)\left[\Theta(z)-\Theta(z^*)\right]>0.$$

Thus, $\Psi(z)>\Phi(z)>0$ for $z<z^*$.

Now suppose $b-c>0$. First we note, that

$$F(b+c,-b+c;c;-z)\overset{Euler\ \ transform}{=}(1+z)^{-b-c}F(b+c,b;c;\frac{z}{1+z})>0$$

Let us rewrite $\Psi(z)$ as

$$\Psi(z)=(Vr+1)(1+z)^{-b}N(z)-1,$$

where

$$N(z)=(1+z)^bF(b,1-b;1-c;-z)-(1+z)^b\frac{(b-c)(b+c+1)}{b(b-1)}\Theta(z^*)z^cF(b+c,1-b+c;c+1;-z)$$

Let us take the derivative of $N(z)$ using (61),(60). By corrolary 7.3 from Appendix D we obtain:

$$\begin{aligned}N'_z(z) &= \frac{b(b-c)}{1-c}(1+z)^{b-1}F(b+1,1-b;2-c;-z)+\\ &+ \Theta(z^*)\frac{(b-c)(b+c-1)c}{b(b-1)}z^c(1+z)^{b-1}F(b+c,-b+c;c;-z)=\\ &= \frac{(b-c)(b+c-1)c}{b(b-1)}z^{1-c}(1+z)^{b-1}F(b+c,-b+c;c;-z)\left[\Theta(z)-\Theta(z^*)\right]<0\end{aligned}$$

Thus, we have proved that $N(z)$ and $(1+z)^{-b}$ are strictly decreasing functions for $z<z^*$. Therefore $\Psi(z)$ is a strictly decreasing function for $z<z^*$. Besides that, $\Psi(0)=Vr>0$ and $\Psi(z^*)=0$.

Thus, $\Psi(z)>0$ for $z<z^*$. △

Proof of the theorem. To prove the theorem we need to check if *the verification properties* hold.

According to the standard technique of stochastic optimal control *the verification properties* are the following:

(A) There exists a function $\tilde{F}=\tilde{F}(x)$ such that for any admissible control $I=I(x)$

$$F(x,I)\le\tilde{F}(x)$$

(B) There exists a control $\tilde{I} = \tilde{I}(x)$ such that

$$F(x, \tilde{I}) = \tilde{F}(x)$$

Let us show that the property (A) holds.

Applying the Ito formula to $\left(e^{-rt}\tilde{F}(X_t)\right)_{t\geq 0}$ we obtain

$$e^{-r(t\wedge\tau)}\tilde{F}(X_{(t\wedge\tau)}) = \tilde{F}(X_0) + \int_0^{t\wedge\tau} e^{-rs}L(I)\tilde{F}(X_s)ds + \int_0^{t\wedge\tau} e^{-rs}\gamma X_s\tilde{F}'(X_s)dW_s + \\ + \int_0^{t\wedge\tau} e^{-rs}\beta\sqrt{I_sX_s}\tilde{F}'(X_s)d\tilde{W}_s$$

Let us notice that by lemma 3.2 we have for any admissible control I

$$IL_1\tilde{F}(x) + L_2\tilde{F}(x) - I = I(L_1\tilde{F}(x) - 1) + L_2\tilde{F}(x) \leq \tilde{I}\left(L_1\tilde{F}(x) - 1\right) + L_2\tilde{F}(x) = 0. \tag{36}$$

Taking the mathematical expectation E_x of $e^{-rt}\tilde{F}(X_t)$, by (36) we obtain

$$\begin{aligned}
\tilde{F}(x) &= \mathsf{E}_x e^{-r(t\wedge\tau)}\tilde{F}(X_{t\wedge\tau}) - \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\left(IL_1 + L_2\right)\tilde{F}(X_s)ds - \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\gamma X_s\tilde{F}'(X_s)dW_s - \\
&- \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\beta\sqrt{I_sX_s}\tilde{F}'(X_s)d\tilde{W}_s \geq \\
&\geq \mathsf{E}_x e^{-r(t\wedge\tau)}\tilde{F}(X_{t\wedge\tau}) + \mathsf{E}_x\int_0^{t\wedge\tau}(-I)e^{-rs}ds - \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\gamma X_s\tilde{F}'(X_s)dW_s \qquad (37)\\
&- \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\beta\sqrt{I_sX_s}\tilde{F}'(X_s)d\tilde{W}_s.
\end{aligned}$$

Note, that the stochastic integrals in (37) are martingales. Therefore the mathematical expectation of those integrals is equal to zero. Letting t go to infinity in (37) we obtain $\tilde{F}(x) \geq F(x, I)$, as $e^{-r(t\wedge\tau)}\tilde{F}(X_{t\wedge\tau}) \to Ve^{-r\tau}$. Indeed, if $\tau < \infty$ then $\tilde{F}(X_\tau) = V$. And if $\tau = \infty$ then $\tilde{F}(X_t)$ is bounded, $e^{-r(t\wedge\tau)} \to 0$ and $Ve^{-r(t\wedge\tau)} \to$, $t \to \infty$. Thus it follows from (37) that $\tilde{F}(x) \geq F(x, I)$, i.e. the property (A) holds.

Let us check the property (B) similary.

Applying Ito formula to $e^{-rt}\tilde{F}(X_t)$ and taking mathematical expectation we have:

$$\begin{aligned}
\tilde{F}(x) &= \mathsf{E}_x e^{-r(t\wedge\tau)}\tilde{F}(X_{t\wedge\tau}) - \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\left(\tilde{I}L_1 + L_2\right)\tilde{F}(X_s)ds - \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\gamma X_s\tilde{F}'(X_s)dW_s - \\
&- \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\beta\sqrt{I_sX_s}\tilde{F}'(X_s)d\tilde{W}_s = \\
&= \mathsf{E}_x e^{-r(t\wedge\tau)}\tilde{F}(X_{t\wedge\tau}) + \mathsf{E}_x\int_0^{t\wedge\tau}(-\tilde{I})e^{-rs}ds - \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\gamma X_s\tilde{F}'(X_s)dW_s - \qquad (38)\\
&- \mathsf{E}_x\int_0^{t\wedge\tau} e^{-rs}\beta\sqrt{I_sX_s}\tilde{F}'(X_s)d\tilde{W}_s
\end{aligned}$$

The mathematical expectation of two last terms in (38) is equal to zero. Thus, as $t \to \infty$ we obtain:

$$\tilde{F}(x) = \mathsf{E}_x\left(\int_0^\tau(-\tilde{I})e^{-rs}ds + Ve^{-r\tau}\right) = F(x, \tilde{I}) \ \triangle\,.$$

4 Appendix A

We have used a number of special functions in this paper. For the reader's convenience we include some definitions and properties of special functions in appendices A, B, C.

A function $I_\nu(z)$ is called a modified Bessel function of first order (see [4], [3]) if

$$I_\nu(z) = (z/2)^\nu \sum_{n=0}^{\infty} \frac{(z^2)/4^n}{n!\Gamma(\nu+n+1)}, \tag{39}$$

A function $K_\nu(z)$ is called a modified Bessel function of the second order if

$$K_\nu(z) = \pi/2 \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}. \tag{40}$$

We use the following properties of Bessel functions:

$$\frac{d}{dz}\left(z^{\frac{\nu}{2}} I_\nu(2\sqrt{z})\right) = z^{\frac{\nu-1}{2}} I_{\nu-1}(2\sqrt{z}) \tag{41}$$

$$\frac{d}{dz}\left(z^{\frac{\nu}{2}} K_\nu(2\sqrt{z})\right) = -z^{\frac{\nu-1}{2}} K_{\nu-1}(2\sqrt{z}) \tag{42}$$

The asymptotic behaviour is given by

$$I_\nu(2\sqrt{z}) \sim z^{\frac{\nu}{2}}/\Gamma(\nu+1), \ (\nu \neq -1, -2, ...) \text{ as } z \to 0, \tag{43}$$

$$K_\nu(2\sqrt{z}) \sim \frac{1}{2}\Gamma(\nu) z^{-\frac{\nu}{2}}, (\nu > 0) \text{ as } asz \to 0. \tag{44}$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \text{ as } z \to \infty, \tag{45}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \text{ as } z \to \infty. \tag{46}$$

5 Appendix B

A function $M(a, b, z)$ is called a confluent hypergeometric function of the first order (see [4], [3]) if

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{47}$$

$(a)_n = a(a+1)..(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$,

A function $U(a, b, z)$ is called a confluent hypergeometric function of the second order if

$$U(a, b, z) = \frac{\pi}{\sin \pi b}\left(\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right). \tag{48}$$

We use the following properties of confluent hypergeometric functions:

Integral representation

$$\Gamma(a)U(a, b, z) = \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt, a > 0, z > 0. \tag{49}$$

Differential relations

$$\frac{d}{dz}\left[e^{-z}z^{b-a}M(a,b,z)\right] = (b-a)e^{-z}z^{b-a-1}M(a-1,b,z) \quad (50)$$

$$\frac{d}{dz}\left[e^{-z}z^{b-a}U(a,b,z)\right] = -e^{-z}z^{b-a-1}U(a-1,b,z). \quad (51)$$

Some identities

$$M(a,b,z) - M(a-1,b,z) = \frac{z}{b}M(a,b+1,z) \quad (52)$$

$$(b-a)U(a,b,z) + U(a-1,b,z) = zU(a,b+1,z). \quad (53)$$

Asymptotical behaviour as $z \to 0$

$$M(a,b,z) \to 1(b \neq n) \quad (54)$$

$$U(a,b,z) = \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b} + O(|z|^{b-2}), (\text{for } b > 2). \quad (55)$$

Asymptotical behaviour as $z \to \infty$

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)}e^{z}z^{a-b}[1 + O(|z|^{-1})], (\text{for } z > 0), \quad (56)$$

$$U(a,b,z) = z^{-a}\left[1 + O(|z|^{-1})\right]. \quad (57)$$

6 Appendix C

A function $F(a,b;c;z)$ is called a hypergeometric function if :

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(a)_n = a(a+1)..(a+n-1)$.

We use the following properties of hypergeometric functions:

Differential relations

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z) \quad (58)$$

$$\frac{d}{dz}\left[z^{c-1}F(a,b;c;z)\right] = (c-1)z^{c-2}F(a,b;c-1;z) \quad (59)$$

$$\frac{d}{dz}\left[(1-z)^{a-c+1}z^{c-1}F(a,b;c;z)\right] = (c-1)z^{c-2}(1-z)^{a-c}F(a,b-1;c-1;z) \quad (60)$$

$$\frac{d}{dz}\left[(1-z)^{a}F(a,b;c;z)\right] = -\frac{a(c-b)}{c}(1-z)^{a-1}F(a+1,b;c+1;z). \quad (61)$$

Some identities

$$zF'_z(a,b;c;z) + aF(a,b;c;z) - aF(a+1,b;c;z) = 0 \quad (62)$$

$$(c-a-1)F(a,b;c;z) + aF(a+1,b;c;z) - (c-1)F(a,b;c-1;z) = 0 \quad (63)$$

$$zF'_z(a,b;c;z) + (c-1)F(a,b;c;z) - (c-1)F(a,b;c-1;z) = 0. \quad (64)$$

Integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, (c > b > 0). \tag{65}$$

Asymptotical behaviour as $z \to 0$

$$F(a,b;c;z) \to 1. \tag{66}$$

Asymptotical behaviour as $z \to \infty$

$$F(a,b;c;z) \sim \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}, a > 0. \tag{67}$$

Euler transform

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right).$$

7 Appendix D

Lemma 7.1 *Let $f \neq 0$ and $u = u(x)$ and $v = v(x)$ be two linearly independent solutions of the second order differential equation*

$$f(x)y_{xx} + g(x)y_x + h(x)y = 0. \tag{68}$$

Then $\Theta(x) = u(x)/v(x)$ is strictly monotone and its derivative is

$$\Theta'_x(x) = \frac{const}{v^2(x)} e^{-\int \frac{g(x)}{f(x)}dx}. \tag{69}$$

Proof. Since u and v are two linearly independents solutions of (68), the following equalities hold:

$$f(x)u_{xx} + g(x)u_x + h(x)u = 0 \tag{70}$$

$$f(x)v_{xx} + g(x)v_x + h(x)v = 0. \tag{71}$$

Multiply (70) by v and (71) by u. By subtracting the second term from the first we obtain

$$f(x)\left(u''v - v''u\right) + g(x)\left(u'v - v'u\right) = 0. \tag{72}$$

Let $w = u'v - v'u$. Then $w' = u''v - v''u$. Therefore we can rewrite (72) as a first order differential equation in separated variables. The function $w(x) = const\ exp\left(-\int \frac{g(x)}{f(x)}dx\right)$ is the solution of this equation. Thus we obtain

$$\Theta'(x) = \frac{u'v - v'u}{v^2} = \frac{const}{v^2}\left[e^{-\int \frac{g(x)}{f(x)}}\right]. \tag{73}$$

△

Corrolary 7.1 *The function $\Theta(z) = -K_{c-1}(2\sqrt{z})/I_{c-1}(2\sqrt{z})$ is strictly increasing and $\Theta'_z(z) = \frac{1}{2z}\left[I_{c-1}(2\sqrt{z})\right]^{-2}$.*

Corrolary 7.2 *For* $b > 1$*, the function* $\Theta(z) = -(b-1)M(b, 2b+1, z^{-1})/2U(b, 2b+1, z^{-1})$ *is strictly increasing and* $\Theta'_z(z) = \frac{1}{2}e^{1/z}z^b\left[(b, 2b+1, z^{-1})\right]^{-2}$.

Corrolary 7.3 *For* $c > 1$ *and* $b > 1$ *the function*
$\Theta(z) = -\frac{(b-1)b^2}{c(c-1)(b+c-1)}z^{1-c}F(b+1, 1-b; 2-c; -z)/F(b+c, -b+c; c; -z)$ *is strictly increasing and*

$$\Theta'_z(z) = \frac{(b-1)b^2}{c(b+c-1)}z^{-c}(1+z)^{-c-1}\left[F(b+c, -b+c; c; -z)\right]^{-2}.$$

References

[1] Boguslavskaya, E. V. Exact solution of a problem of the optimal control of investments in a diffusion model., Russian Math. Surveys 52 (1997), no.2, 396-397

[2] Dubins, L.E.; Shepp, L. A.; Shiryaev, A. N. Optimal stopping rules and maximal inequalities for Bessel processes., Theory Probab. Appl. 38 (1993), no.2, 226-261

[3] Erdelyi, A.; Magnus, W.; Oberhettinger, F; Tricomi, F. G. Higher transcendental functions. Vol. I. Based on notes left by Harry Bateman. Reprint of the 1955 original. Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. 302pp.

[4] Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. / Ed. by M. Abramowitz and I. A. Stegun, National Bureau of Standards. 10th print. New York: Wiley-Inerscience, 1972, 1046 p.

[5] Kramkov, D. O.; Mordetski, E. An integral option. Theory Probab. Appl. 39(1994), no.1, 162-172(1995)

[6] R. S. Pindyck. Investments of uncertain cost. Journal of Financial Economics. 34 (1993) p.53-76. North Holland.

[7] Polyanin, A.D.; Zaitsev, V.F. Handbook of exact solutions for ordinary differential equations. CRC Press, Boca Raton, FL, 1995.xii+707pp.

[8] Zhanblan -Pike, M.(Jeanblanc-Picque, M.); Shiryaev, A. N. Optimization of the flow of dividents. Russian Math. Surveys 50 (1995), no.2, 257-277