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CODING THEORY

SECTIONS OF THE DEL PEZZO SURFACES AND GENERALIZED WEIGHTS[1]

M. I. Boguslavsky UDC 621.391.1

We analyze sections of split Del Pezzo surfaces and compute generalized Hamming weights of the corresponding algebraic-geometric codes.

1. Introduction: Hamming weights and projective systems

Generalized Hamming weights (or the weight hierarchy) were first introduced in [1]. The generalized weights of a linear $[n,k,d]_q$ code are a monotonic ordered set of k integers $d_1 = d < d_2 < \ldots < d_{k-1} < d_k = n$. The first weight d_1 is equal to the minimum distance, and the last weight d_k is equal to the length of the code. Generalized weights describe the performance of the code if it is used as a cryptographic code in channels of a certain type; they also have some other applications. Further references may be found in the survey [2].

Definition. The *support* $\chi(D)$ of a code D is defined as the set of coordinate positions such that there exists a codeword that has a nonzero bit in this position, i.e., $\chi(D) = \{i : \exists(x_1, x_2, \ldots, x_n) \in D : x_i \neq 0\}$.

Definition. The *r-th generalized Hamming weight* of a linear code C is the minimal support size of an r-dimensional subcode of C. The set of all generalized weights is called the *weight hierarchy*.

Generalized weights admit a natural geometric interpretation. As is shown in [3], the study of nondegenerate linear q-ary $[n,k]$ codes may be reduced to the study of nondegenerate projective systems, that is, of n-point posets whose elements are points in a $(k-1)$-dimensional projective space over the finite field $\mathbb{F}_q$. In the sequel, all codes are assumed to be nondegenerate, i.e., with the effective length equal to the length. Projective systems are also assumed to be nondegenerate, i.e., not contained in a hyperplane.

By $|X|$, we denote the cardinality of a finite set X and the number of $\mathbb{F}_q$-points of an algebraic set X (it will always be clear what q is considered).

We shall call two systems equivalent if one may be obtained from the other by a projective transformation; we shall call two codes equivalent if one may be obtained from the other by a permutation of coordinates and by multiplication of some coordinates by scalars. The following construction gives a bijective map between the set of classes of equivalent codes and the set of classes of equivalent projective systems.

For a given linear $[n,k]_q$ code C, the n coordinate functions may be considered as elements of the dual $\mathbb{F}_q$-linear k-dimensional space C^*. The images of these points under a projectivization $C^* \setminus \{0\} \to \mathbb{P}^{k-1}$ form a projective system.

On the other hand, given a projective system $X \subset \mathbb{P}^{k-1}$, we may lift it to a poset $\{x_1, \ldots, x_n\}$ in a k-dimensional vector space V and define a code C as the image of V^* under the map $V^* \to \mathbb{F}_q^n$ given by $v \mapsto (x_1(v), \ldots, x_n(v))$.

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TABLE 1. Codes from the Del Pezzo Surfaces

	n	k	d
$\mathcal{D}^9$	q^2+q+1	10	q^2-2q
$\mathcal{D}^8$	q^2+2q+1	9	q^2-2q
$\mathcal{D}^7$	q^2+3q+1	8	q^2-2q
$\mathcal{D}^6$	q^2+4q+1	7	q^2-2q+1
$\mathcal{D}^5$	q^2+5q+1	6	q^2
$\mathcal{D}^4$	q^2+6q+1	5	q^2+2q
$\mathcal{D}^3$	q^2+7q+1	4	q^2+4q+1

One may check that these maps are inversions of each other. The minimum distance of a code is equal to the minimum number of points of the system outside a hyperplane and the rth generalized weight is equal to the minimum number of points outside a linear space of codimension r, i.e.,

$$d_r = \min_{\Pi^r}(|X| - |X \cap \Pi^r|), \tag{1}$$

where the minimum is taken over all linear spaces Π^r of codimension r in $\mathbb{P}^{k-1}$. We call the spaces at which the minimum is attained the *maximal sections* of X; they contain the maximum possible number of points of X.

Maximal sections satisfy the following monotonicity property: The number of points in a maximal section by a space of codimension r is greater than the number of points in a maximal section of codimension $r+1$.

The sets of $\mathbb{F}_q$-points of algebraic varieties are a good source of projective systems (see [3]). Hirschfeld, Tsfasman and Vlăduţ [4] computed generalized weights for codes constructed from Hermitian varieties, Nogin [5] and Wan [6] independently computed them for codes from multidimensional quadrics, and Nogin [7] also computed generalized weights for codes from Grassmann varieties. The problem of weights for codes on Veronese varieties is discussed in [8]. Weights for classical Reed–Muller codes (they correspond to $\mathbb{F}_2$-points of affine Veronese varieties) are computed by Wei [9], and Heijnen and Pellikaan [10] computed the weights for affine Veronese varieties over an arbitrary field.

We compute the weight hierarchy for algebraic-geometric codes constructed from Del Pezzo surfaces. The classical parameters of the corresponding q-ary ($q \geq 3$) codes listed in Table 1 are computed in Sec. 4. Generalized weights are given in Table 2. In Sec. 2, we discuss the properties of the Del Pezzo surfaces over finite fields; in Sec. 3, we develop the necessary geometric theory; and in Sec. 4, we compute the generalized weights.

2. Del Pezzo surfaces

The family of Del Pezzo surfaces was studied in the classical papers of the nineteenth and early twentieth centuries (see [11] and the bibliography therein). In modern terms, a *Del Pezzo surface* is an algebraic surface with ample anticanonical divisor class. A thorough modern treatment of the Del Pezzo surfaces can be found in [12]. In the present paper, we restrict ourselves to birationally trivial Del Pezzo surfaces of degrees 3 to 9 split over $\mathbb{F}_q$. A surface of degree k is denoted by $\mathcal{D}^k$. The family considered is interesting for several reasons. On one hand, it is the simplest family of surfaces birational to $\mathbb{P}^2$; on the other hand, these surfaces are a natural generalization of both a cubic surface in $\mathbb{P}^3$ (which coincides with $\mathcal{D}^3$) and a Veronese surface in $\mathbb{P}^9$ (which coincides with $\mathcal{D}^9$). Note that algebraic-geometric codes from Veronese varieties are the q-ary projective Reed–Muller codes (see [13]).

The family of surfaces considered is constructed as follows. Take a projective plane and choose, whenever possible, ℓ ($\ell = 0, \ldots, 6$) $\mathbb{F}_q$-points in the general position. The blow-up of the plane at these points is a birationally trivial surface S with a map $\pi : S \to \mathbb{P}^2$ contracting the exceptional divisors $E_1, \ldots, E_\ell$. We say that ℓ ($\ell \leq 6$) points in $\mathbb{P}^2$ are in the *general position* if no three of them are collinear and no six of them are contained in a quadric. Let H denote the divisor class of a line in $\mathbb{P}^2$. Define a divisor class $\widetilde{C}$ on S by

$$\tilde{C} = \pi^* 3H - E_1 - \ldots - E_\ell.$$

Consider the complete linear series $|\tilde{C}|$ on S. This series gives an embedding $\xi : S \to \mathbb{P}^k$ ($k = 9 - \ell$). The image of this map is a surface of degree k.

We shall study hyperplane sections of these surfaces. All exceptional divisors on these surfaces are defined over $\mathbb{F}_q$. For small values of q, it may happen that there are not enough $\mathbb{F}_q$-points in the general position in $\mathbb{P}^2$. This problem has been studied by Hirshfeld in [14]. He proves that all seven surfaces $\mathcal{D}^3$, $\mathcal{D}^4$, $\ldots$, $\mathcal{D}^9$ exist iff $q > 4$, and six surfaces $\mathcal{D}^4$, $\mathcal{D}^5$, $\ldots$, $\mathcal{D}^9$ exist iff $q > 3$. In the sequel, we assume that $q > 3$. Without this assumption, we would obtain very poor projective systems; because of syzygies like $x_0^3 = x_0$ on the values of monomials (giving the Veronese embedding) at $\mathbb{F}_q$-points, all $\mathbb{F}_q$-points of $\mathcal{D}^k$ would be contained in a hyperplane.

Whenever we consider $\mathcal{D}^3$, we assume that $q > 4$.

All Del Pezzo surfaces of a given degree $k = 5, 6, \ldots, 9$ are isomorphic over $\mathbb{F}_q$, i.e., they do not depend on the choice of blow-up centers (see [12]). Surfaces of degrees 3 and 4 depend on the choice of centers. However, in most cases, the corresponding projective systems have the same parameters n, k, $d_1, \ldots, d_k$. The only exception are surfaces of degree 3; generalized weights for the corresponding codes depend on the existence of the so-called *Eckardt points*. By definition, an *Eckardt point* on a surface is a point of intersection of three lines lying on the surface (see [14] and [11]). A general Del Pezzo surface of degree 3 has no Eckardt points. As we shall show later, codes corresponding to surfaces with Eckardt points are worse.

Lemma 1. *The number of $\mathbb{F}_q$-points on $\mathcal{D}^k$ ($k = 3, \ldots, 9$) is equal to $q^2 + (10 - k)q + 1$.*

Proof. The projective plane $\mathbb{P}^2$ has $q^2 + q + 1$ $\mathbb{F}_q$-points. A surface $\mathcal{D}^k$ is a blow-up of $\mathbb{P}^2$ at $9 - k$ $\mathbb{F}_q$-points. A blow-up at each point replaces an $\mathbb{F}_q$-point by a line defined over $\mathbb{F}_q$, so the number of $\mathbb{F}_q$-points increases by q. $\triangle$

3. Plan of calculations

To find the minimum (1), we need to compute for each $k = 3, 4, \ldots, 9$ and each $r = 1, 2, \ldots, k$ the number

$$\mathcal{M}(k, r) = \max_{\Pi^r} |\Pi^r \cap \mathcal{D}^k|, \tag{2}$$

i.e., the maximum possible number of points on a section of the Del Pezzo surface $\mathcal{D}^k$ by a space Π^r of codimension r in $\mathbb{P}^k$.

Let $\mathcal{L}$ be a linear series on a surface. The *base locus* of $\mathcal{L}$ is the intersection of supports of all divisors in $\mathcal{L}$. It is an algebraic set with components of dimensions zero and one with some additional structure. In particular, we may correctly define a blow-up of a base locus.

Let $\tilde{X}$ be the proper preimage of an intersection of cubics $X = C_1 \cap \ldots \cap C_r$ under the blow-up π^{-1}, i.e., $\tilde{X} = \overline{\pi^{-1}(X) \setminus E}$ (see [15] or [16]). The base locus of the blow-up $\tilde{\mathcal{L}}$ of a linear system $\mathcal{L}$ will be called the *blow-up of the base locus* X of a linear system $\mathcal{L}$. Thus, the blow-up of X coincides with $\tilde{X} + (\mathrm{mult}_P(C) - 1)E$, where $\mathrm{mult}_P(C)$ is the multiplicity at P of a general curve C from $\mathcal{L}$. By an abuse of notation, we denote the blow-up of the intersection $C_1 \cap \ldots \cap C_r$ by $\pi^{-1}(C_1 \cap \ldots \cap C_r)$.

We consider a Del Pezzo surface $\mathcal{D}^k$ in the embedding given by the linear series

$$\tilde{C} = \pi^* 3H - E_1 - \ldots - E_\ell,$$

where E_i are the exceptional divisors of blow-ups, H is the class of a line in $\mathbb{P}^2$, $\ell = 9 - k$ is the number of blow-ups. Thus, any hyperplane section of $\mathcal{D}^k$ has the form

$$\pi^* C - E_1 - \ldots - E_\ell,$$

where C is a plane cubic containing the blow-up centers $p_1, \ldots, p_\ell$, and to each cubic corresponds a hyperplane section of $\mathcal{D}^k$. An exceptional divisor E_i is contained in a hyperplane section iff the corresponding plane cubic has multiplicity at least two at p_i. Since a subspace of codimension r is the intersection of r linearly independent hyperplanes, a section of $\mathcal{D}^k$ by a subspace of codimension r corresponds to the intersection of r independent (as elements of $|3H| \simeq \mathbb{P}^9$) plane cubics passing through the blow-up centers.

Thus, the $\mathbb{F}_q$-points on a section of $\mathcal{D}^k$ by a space of codimension r are the same as the $\mathbb{F}_q$-points of the blow-up at $9-k$ points of the intersection of r independent plane cubics passing through $9-k$ points in the general position. Therefore, for each $k = 3, 4, \ldots, 9$ and each $r = 1, 2, \ldots, k$, the maximum (2) coincides with

$$\mathcal{M}(k, r) = \max_{C_1, \ldots, C_r} |\pi^{-1}(C_1 \cap \ldots \cap C_r)|, \tag{3}$$

where the maximum is taken over all sets of r linearly independent cubics $C_1, C_2, \ldots, C_r$ passing through fixed $9 - k$ $\mathbb{F}_q$-points in the general position. As we shall see, in all cases but one, this number does not depend on the choice of these $9 - k$ points.

To compute the maximum (3), we search through the intersections of plane cubics. To each intersection of plane cubics, we assign a certain *type*, which is determined by the degrees of components and their mutual position. For each intersection type, we compute

(a) the dimension of the linear series of cubics passing through this intersection, i.e., the maximum possible number of independent cubics each of which contains this intersection;

(b) the existence of multiple points. These points have the following property: The blow-up $\pi^{-1}(C_1 \cap \ldots \cap C_r)$ at such a point P of the intersection $C_1 \cap \ldots \cap C_r$ will contain the exceptional divisor;

(c) how many $\mathbb{F}_q$-points in the general position exist on intersections of this type;

(d) how many $\mathbb{F}_q$-points in the general position exist such that some of them coincide with multiple points;

(e) the maximum possible number of $\mathbb{F}_q$-points on an intersection of this type and on its blow-ups.

Thus we obtain Table 2. The dimension in (a) determines the codimension of the section of $\mathcal{D}^k$ corresponding to this intersection. The number in (c) determines the surfaces that have a section corresponding to the intersection (if there exist less than $9 - k$ $\mathbb{F}_q$-points in the general position, then this intersection does not correspond to any section of $\mathcal{D}^k$, $k = 3, \ldots, 8$). The number in (e) determines the generalized weights, and numbers in (b) and (d) are necessary for the computation of (e).

Then, for each $k = 3, 4, \ldots, 9$ and each $r = 1, 2, \ldots, k$, we choose from Table 2 all intersection types corresponding to sections of $\mathcal{D}^k$ by spaces of codimension r and find among them the one with the maximum number of $\mathbb{F}_q$-points. This number of points gives the maximum in (3) as well as in (2).

In the next section, we give some geometric facts that we need for the computation of the dimension of the linear series of cubics passing through a given set in the plane. The main result is Theorem 1.

4. Dimensions of linear series

An intersection X of a linear series of plane cubics consists of an effective divisor (a curve not necessarily reduced or irreducible) M of degree $d = 0, \ldots, 3$ and a 0-subscheme Γ, which is the intersection of divisors $C - M$, where C is a cubic from the series. Usually, the most of the $\mathbb{F}_q$-points of X lie in M. By $\mathcal{O}(n)$, we denote the complete linear series of all curves of degree n on the plane; by $\mathcal{J}_X(\mathcal{L})$, we denote the subsheaf corresponding to all curves passing through an algebraic set X; $\mathcal{J}_X(\mathcal{O}(n))$ is abbreviated as $\mathcal{J}_X(n)$.

Proposition 1. *Any divisor D from $\mathcal{J}_X(n)$ contains M as its component, i.e., $D - M \geq 0$.*

This proposition is obvious. The divisor M is called the *fixed component* of $\mathcal{J}_X(\mathrm{L})$, and the points from Γ are called the *base points.* Denote by d the degree of the fixed component. Clearly, $d = 0, \ldots, 3$.

The following proposition allows us to separate the conditions imposed on the series by the fixed component from the conditions imposed by the base points.

Proposition 2. *The map $\mathcal{J}_X(3) \to \mathcal{O}(3-d)$ given by $D \mapsto D - M$ is an injective map onto $\mathcal{J}_\Gamma(3-d)$.*

This proposition is also evident.

Thus, we need only compute how many independent conditions a 0-subscheme Γ imposes on $\mathcal{O}(n)$, $n = 1, 2, 3$.

The quantity

$$h^0\left(\mathcal{J}_\Gamma(\mathcal{L})\right) - \left(h^0(\mathcal{L}) - \deg \Gamma\right) \tag{4}$$

is called the *redundancy* of Γ with respect to $\mathcal{L}$. Roughly speaking, zero redundancy corresponds to independence of the conditions imposed by different points of Γ.

Proposition 3. *The redundancy of Γ with respect to a linear series $\mathcal{L}$ equals $h^1(\mathcal{J}_\Gamma(\mathcal{L}))$.*

This proposition is a corollary of the Riemann–Roch theorem; see [15].

Let us a define for each 0-subscheme Γ in $\mathbb{P}^2$ and each linear series $\mathcal{L}$ a class of 0-subschemes Δ that are in some sense dual to Γ. Let C and C' be general curves from $\mathcal{L}$. Their intersection consists of a curve M and a 0-subscheme $\Delta + \Gamma$. It is clear that Δ depends on the choice of C and C', but this dependence will not be important for our purposes.

We use the reciprocity formula II [15, p. 716] to compute $h^1(\mathcal{J}_\Gamma(n))$. In [15], it is assumed that the ground field is $\mathbb{C}$. However, the proof given there is valid for an arbitrary algebraically closed field. We shall use the simplest form of this formula that is valid for regular surfaces. Since regularity is invariant under birational transforms and $\mathbb{P}^2$ is regular, all surfaces considered in this work are regular.

In the notation introduced above, this formula reads

$$h^1(\mathcal{J}_\Gamma(n)) = h^0(\mathcal{J}_\Delta(\mathcal{O}(K + nH)), \tag{5}$$

where K is the canonical class of the surface, H is the class of a hyperplane section, and $\mathcal{J}_\Delta(\mathcal{O}(K+nH)$ is the subsheaf of $\mathcal{O}(K+nH)$ that corresponds to the divisors passing through Δ. Thus, the redundancy $h^1(\mathcal{J}_\Gamma(n))$ is equal to the dimension of a certain linear series passing through Δ. For the projective plane, $K = -3H$; so $h^0(\mathcal{J}_\Delta(\mathcal{O}(K+nH)) = h^0(\mathcal{J}_\Delta(n-3))$. This number is equal to the dimension of the linear series of all curves of degree $n-3$ passing through Δ.

Substituting the invariants of $\mathbb{P}^2$ into (5), we get the following proposition.

Proposition 4. *Let Γ be a 0-subscheme of $\mathbb{P}^2$. Assume that the linear series $\mathcal{J}_\Gamma(n)$ does not have a fixed component. Then the redundancy of Γ with respect to $\mathcal{L}$ satisfies the equality*

$$h^1(\mathcal{J}_\Gamma(\mathcal{L})) = h^0(\mathcal{J}_\Delta(\mathcal{L} - 3H)).$$

The following theorem gives the dimension of the linear series of all plane cubics passing through a given intersection of cubics.

Theorem 1. *Let X be the intersection of cubics in $\mathbb{P}^2$ consisting of a curve M of degree D and a 0-scheme Γ of degree d; then either*

$$h^0(\mathcal{J}_X(3)) = \frac{(5-d)(4-d)}{2} - s$$

or $d = 0$, $s = 9$, and $h^0(\mathcal{J}_X(3)) = 2$.

Proof. Consider the map $\mathcal{J}_X(3) \to \mathcal{O}(3-d)$ given by $C \mapsto C - M$. Clearly, this map is injective. Since $D \in \mathcal{J}_\Gamma(3-d)$ implies $D + M \in \mathcal{J}_X(3)$, the image of this map coincides with $\mathcal{J}_\Gamma(3-d)$.

We need to compute $h^0(\mathcal{J}_X(3)) = h^0(\mathcal{J}_\Gamma(3-d))$. By the definition of redundancy,

$$h^0(\mathcal{J}_\Gamma(n)) = h^0(\mathcal{O}(n)) - \deg \Gamma + h^1(\mathcal{J}_\Gamma(n)).$$

Thus,

$$h^0(\mathcal{J}_X(3)) = h^0(\mathcal{O}(3-d)) - s + h^1(\mathcal{J}_\Gamma(3-d)). \tag{6}$$

From Proposition 4 it follows that $h^1(\mathcal{J}_\Gamma(3-d)) = h^0(\mathcal{J}_\Delta(-d))$. Therefore, the redundancy equals one for $d = 0$, $\Delta = \varnothing$ and equals zero otherwise. If $d = 0$ and $\Delta = \varnothing$, then $s = 9$, and from (6) we get that $h^0(\mathcal{J}_X(3)) = 2$. Note that this is equivalent to the classical result stating that a plane cubic containing eight points of intersection of other two cubics, contains all nine points of their intersection.

If the redundancy equals zero, the conditions imposed on $\mathcal{O}(3-d)$ by the points of Γ are independent, so the number of conditions s cannot exceed $h^0(\mathcal{O}(3-d))$. Therefore, $s \leq h^0(\mathcal{O}(3-d)) = \frac{(4-d)(5-d)}{2}$.

Since the conditions are independent, it follows from (6) that $h^0(\mathcal{J}_X(3)) = \frac{(5-d)(4-d)}{2} - s$. $\triangle$

Remark 1. Let $\mathcal{L}$ be a linear series on the projective plane consisting of curves of degree n, $n = 1, 2, 3$. Let Γ be its base locus, and let P_1 be a simple point of Γ. Consider the linear series $\mathcal{L}' \subseteq \mathcal{L}$ of all curves with a prescribed tangent at P_1. Applying the reciprocity formula to a blow-up of $\mathbb{P}^2$ at P_1, we get (see [15]) that $\mathcal{L}'$ is strictly contained in $\mathcal{L}$, i.e., a specified tangent is a condition independent of conditions imposed by other points of Γ. In particular, if $\mathcal{L}$ is the linear series of all curves of a given degree passing through Γ, $\mathcal{L} = \mathcal{J}_\Gamma(n)$, $n = 1, 2, 3$, then the blow-up of $\mathcal{L}$ at any simple point P_1 of Γ does not have a base point on the exceptional divisor corresponding to P_1.

5. Intersections of cubics

First, we determine the possible degrees of a one-dimensional component M and a 0-dimensional component Γ. The *intersection type* will be determined by these degrees. Using Theorem 1, we get that the dimension of the linear series of cubics passing through this intersection depends only on these degrees.

Then, for each intersection type, we determine the maximum possible number of $\mathbb{F}_q$-points in the general position on an intersection of this type (this will determine whether this intersection corresponds to a section of $\mathcal{D}^3$, $\mathcal{D}^4$, $\mathcal{D}^5$, $\mathcal{D}^6$, $\mathcal{D}^7$, or $\mathcal{D}^8$) and how many multiple points in the general position it has (this determines how many exceptional divisors contain the corresponding section).

Finally, we compute how many $\mathbb{F}_q$-points can be contained in an intersection of a given type and its blow-ups.

The results are given in Table 2.

5.1. Enumeration of intersections. (**I**). Let us assume that the degree of M is equal to three. It is clear that there exists a unique cubic passing through M, and the dimension of $\mathcal{J}_X(3)$ is equal to 0, so X imposes 9 independent conditions on $\mathcal{O}(3)$.

If X is the sum of three lines, we may choose six points in the general position on X. If all three lines are concurrent, X has a unique multiple point. In the general position, we may choose either this multiple point and 2 simple points, or 6 simple points. In the converse case, one of the lines contains more than two points, which is impossible by the definition of the general position.

Similarly, if the three lines are not concurrent, then in a general position we may choose either three multiple points, or two multiple points and two simple points, or one multiple point and four simple points, or six simple points.

Note that it may happen that for given six points $P_1, P_2, \ldots, P_6$ in the general position, there do not exist three concurrent lines passing through these points. If three such lines exist, the Del Pezzo surface corresponding to these points $P_1, P_2, \ldots, P_6$ has an *Eckardt point*, and the point of intersection of these three lines is mapped to the Eckardt point.

It can be easily checked that this is the only case where the properties of intersection depend on the choice of a specific surface of the given degree.

The case where X is a sum of a quadric and a line is similar.

Clearly, for $q > 4$, there exists an irreducible cubic that contains six $\mathbb{F}_q$-points in the general position, and for $q > 3$, there exists a cubic with five $\mathbb{F}_q$-points in the general position. Indeed, from Theorem 1 it follows that the dimension of the linear series of all cubics passing through these points is positive whenever there exist five or six $\mathbb{F}_q$-points in the general position on the plane.

(**II**). Assume that the degree of M is equal to two. Then Γ cannot contain more than one point. If Γ is empty, our series may be identified with the linear series $\mathcal{O}(1)$ and has dimension three. If Γ consists of a single point P, it follows from Proposition 2 that our series may be identified with the linear series of lines passing through P and has dimension two. Note that if P belongs to M, then each curve from $\mathcal{J}_X(3)$ has a double point at P.

If M is the sum of two lines, we may choose on M one double point and two simple points, or four simple points. If M is an irreducible quadric, there are no multiple points on M, but we may choose five simple points in the general position.

Assume now that the degree of Γ equals one. Then Γ consists of a unique point P. If P belongs to M, the intersection has one multiple point more than M and the same number of points in the general position as M. Otherwise, it has the same number of multiple points and one point more in the general position.

(**III**). Assume that the degree of M equals one. Since $\mathcal{J}_X$ may be embedded in $\mathcal{O}(3)$, Γ may consist of $s = 0, 1, \ldots, 4$ points counted with multiplicities. From Theorem 1 it follows that $h^0(\mathcal{J}_X) = 6 - s$.

The exceptional divisor of the blow-up may be contained in the section if the blow-up center is either a joint point of Γ and M or a point P of Γ such that any curve of $\mathcal{J}_X(3)$ has multiplicity at least two at P. In the latter case, the multiplicity of such a point will be at least four. In Table 2, we denote such points by the symbol $\times$.

Thus, in the general position on X, we may choose either $2+s$ simple points, or (for $s \geq 2$) one multiple and s simple points, or (for $s \geq 2$) two multiple and $s - 2$ simple points, or (for $s = 4$) one multiple and two simple points.

(**IV**). If there is no fixed component, then Γ consists of $s \leq 9$ points.

If Γ is empty, then the dimension of the linear series equals 10. From Theorem 1 it follows that, for $s < 8$, conditions from different points are independent and the dimension equals $10 - s$ for $s < 8$. Any cubic passing through eight points of intersection of two cubics passes through the ninth point of their intersection, which will be defined over the same field as these eight points. Thus, the intersection cannot consist of eight points, and $s > 7$ implies that $s = 9$ and the dimension of the series equals two.

Multiple points of X have degree at least four. It is clear that we may choose s simple points in the general position, or (for $s \geq 4$) one multiple point and $s-4$ simple points, or (for $s \geq 8$) two multiple points and $s - 8$ simple points.

5.2. The number of $\mathbb{F}_q$-points. In this section, we find for each intersection type the maximum possible number of $\mathbb{F}_q$-points on an intersection of this type and on its blow-ups. From the Hasse–Weil theorem it follows that an irreducible plane cubic cannot have more than $q + 2\sqrt{q} + 1$ $\mathbb{F}_q$-points.[2] A conic has at most $q + 1$ $\mathbb{F}_q$-points (and exactly $q + 1$ if it is rational over $\mathbb{F}_q$). A line defined over $\mathbb{F}_q$ has $q + 1$ points. Thus, the sum of a line and a conic has at most $2q + 2$ points. The sum of two lines has at most $2q + 2$ points, and the sum of three lines has at most $3q + 1$ points. The latter number of points is attained if these lines are concurrent. In the converse case, the sum has $3q$ points.

Since we are looking for the intersections with the maximal number of points, we may assume that all components are defined over $\mathbb{F}_q$.

(**I**). If X is a curve of degree three, then it is either an irreducible cubic, or a sum of a quadric and a line, or the sum of three lines. Since $3q > q+2\sqrt{q}+1$ and $3q > 2q+2$ for any q considered, the sum of three lines has more points than an irreducible cubic or the sum of a quadric and a line. Thus, the maximum possible number of points equals $3q + 1$ and is attained at a sum of three distinct concurrent lines. A blow-up of such an intersection at one point may contain at most $4q+1$ points. This number is attained if the blow-up center coincides with the point of intersection of the three lines. This number is the maximum possible since one blow-up cannot add more than q points to the intersection. If we blow up the three lines at two points, we get at most $5q$ points. This number is attained if we blow up a sum of three lines in the general position at two intersection points. Since we add in this way $2q$ points to an intersection that has one point less than a maximal one, and to a maximal one we may add only q points by two blow-ups, this number is the maximum possible. Similarly, the maximum possible number of points on a blow-up at three points equals $6q$ and is attained if the blow-up centers are chosen at three intersection points of three lines in the general position.

[2] This is not the best known bound. However, it is sufficient for our purposes.

We cannot choose four multiple points on a cubic in the general position. This is evident for a reducible cubic; for an irreducible one, note that a line passing through two multiple points has intersection multiplicity with the cubic at least four, so an irreducible cubic has at most one multiple point. Moreover, if the three blow-up centers are at three intersection points of three lines, we cannot choose the fourth blow-up center on any of these lines. Arguing as above, we see that the maximum possible number of points on a blow-up of a cubic at four points is equal to $5q$ and is attained at the same sum of three lines as above; two blow-up centers are located at the points of intersection of the first line with the second and the third one, and two other blow-up centers are on the second and third line outside their intersection point. Similarly, the maximum possible number of points on a blow-up at five points is equal to $4q + 1$.

For $\mathcal{D}^3$, the maximum possible number of points depends on the existence of an Eckardt point. If we can choose three disjoint pairs of blow-up centers (P_1, P_2), (P_3, P_4), (P_5, P_6) so that the lines P_1P_2, P_3P_4, and P_5P_6 are concurrent, then the maximum possible number of points is attained at this configuration and equals $3q + 1$. In the converse case, the maximum is also attained at the sum of three lines passing through the blow-up centers, but is equal to $3q$.

(**II**). Now, let us consider intersections that consist of a quadric and a point (the second row of Table 2). It is clear that the maximum possible number of points on such an intersection is equal to $2q + 2$ and is attained if M is a sum of two lines and the point P is outside any of these lines. The maximum possible number of points on a blow-up at one point is equal to $3q+2$ and is attained if the blow-up center coincides with the intersection point of the lines. If the blow-up center coincides with P and P belongs to one of the lines, there are only $3q + 1$ points.

It is readily seen that the maximum possible number of points on a blow-up at two points is attained if P belongs to one of the lines and one blow-up center coincides with P while the other coincides with the intersection point of the lines. The case of a blow-up at three points is similar. The maximal number of points on a blow-up at four points is attained if P does not belong to any of the lines, one of the blow-up centers coincides with P, and the other three belong to the lines, one of them being the intersection point of the lines. The only way to choose five points in the general position so that one of them coincides with a multiple point on an intersection of this type is to take an intersection with an irreducible conic as a fixed component and the point P belonging to this conic. The blow-up at five points has $2q + 1$ points. We get the same number of points if four blow-up centers are chosen on two lines.

The intersections from the third row of Table 2 are considered similarly.

Remark 2. Intersections with multiple points in Γ are never maximal for $\mathcal{D}^9$. Indeed, from Theorem 1 it follows that a multiple point imposes at least as many conditions as a simple one, so the intersections with multiple points in Γ are never maximal. However, the situation is different for the Del Pezzo surfaces of smaller degrees. Maximal sections for the Del Pezzo surfaces of degrees $3, 4, \ldots, 8$ usually correspond to intersections with blow-up centers in multiple points of Γ.

(**III**). The number of points on an intersection consisting of a line and a 0-subscheme of degree s does not exceed $q + s + 1$. If $s = 0$, the number of points on a blow-up at one or two points is the same; if $s > 0$, the maximum possible number of points on a blow-up at one point is equal to $2q + s$ and is attained if one of the points of Γ belongs to the line and coincides with the blow-up center. The other cases are considered similarly.

(**IV**). Finally, consider the intersection without a fixed component. The number of $\mathbb{F}_q$-points in Γ does not exceed $s = \deg \Gamma$. If an exceptional divisor is contained in the section, then Γ has multiplicity at least 4 at a blow-up center P (in this case, each curve of $\mathcal{J}_X(3)$ has multiplicity at least two at P). If a blow-up center is a simple point of Γ, then it follows from Remark 1 that the image of this point of intersection is empty.

The remaining calculations are trivial.

6. The tables

Table 2 summarizes the results of the previous section. The first column gives the codimension r of a section. Recall that this number is equal to the number of independent cubics passing through this intersection, and $10 - r$ is equal to the number of independent conditions that the intersection imposes on $\mathcal{O}(3)$.

The second column gives the degree d of the one-dimensional component. The third column describes the intersection as an algebraic set. By the symbol $\cdot$, we denote a simple point, and by the symbol $\times$, we denote a point that is a double point for any curve of the series.

The next seven columns give the maximum possible number of points on a blow up of the intersection in $\ell = 0, 1, \ldots, 6$ points in the general position. Recall that $\ell = 10 - k$, where k is the degree of the corresponding Del Pezzo surface.

The value $3q$ at the intersection of the last column with the first row corresponds to a surface without an Eckardt point. As was mentioned above, the corresponding number for a surface with an Eckardt point is equal to $3q + 1$.

Note that the empty cells of Table 2 correspond to the intersections that do not have enough $\mathbb{F}_q$-points in the general position. The zeros correspond to the intersections that have enough points in the general position, but the blow-up at these points is empty.

Table 3 gives the weight hierarchy for the family of codes constructed from the Del Pezzo surfaces. It is obtained from Table 2 by finding for each surface $\mathcal{D}^k$ ($k = 9 - \ell$, $\ell = 0, \ldots, 6$) and each r a cell in Table 2 with the maximal value.

The first row of Table 3 gives the surface. The degree of the surface is one more than the dimension of the corresponding code. The second row gives the number of $\mathbb{F}_q$-points on a corresponding surface; it is equal to the length of the code. The first column gives the number of a generalized weight. We have some empty cells at the lower-right corner of the table; the reason is that a Del Pezzo surface of degree $k - 1$ is embedded in a $(k-1)$-dimensional projective space and the corresponding code has only k weights. The last, kth weight is always equal to the code length n.

To give an example, let us find the fourth weight for the code on $\mathcal{D}^6$. The fourth weight ($r = 4$) corresponds to the seventh row from the top and to the eighth and ninth rows from the bottom in Table 2. The surface $\mathcal{D}^6$ corresponds to a blow-up at three points. Choose the cells in the three chosen rows that correspond to the blow-up at three points. Among these three cells, the one in the seventh row has the maximal value. Subtracting this number from the number of $\mathbb{F}_q$-points on $\mathcal{D}^6$, we get $(q^2+4q+1)-(2q+1) = q^2 + 2q$. Thus, the fourth weight equals $q^2 + 2q$.

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TABLE 2. Intersections of Plane Cubics

<table>
<tr><th>r</th><th>d</th><th>intersection</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr>
<tr><td>1</td><td>3</td><td>cubic</td><td>$3q+1$</td><td>$4q+1$</td><td>$5q+1$</td><td>6q</td><td>$5q+1$</td><td>$4q+1$</td><td>$3q$</td></tr>
<tr><td>2</td><td>2</td><td>quadric and ·</td><td>$2q+2$</td><td>$3q+2$</td><td>$4q+1$</td><td>$4q+1$</td><td>$3q+1$</td><td>$2q+1$</td><td>$q+1$</td></tr>
<tr><td>3</td><td>2</td><td>quadric</td><td>$2q+1$</td><td colspan="3">$3q+1$</td><td>$2q+1$</td><td>$q+1$</td><td></td></tr>
<tr><td>2</td><td>1</td><td>line and 4·</td><td>$q+5$</td><td>$2q+4$</td><td>$3q+3$</td><td>$3q+2$</td><td>$3q+1$</td><td>$2q+1$</td><td>$q+1$</td></tr>
<tr><td>2</td><td>1</td><td>line and ×</td><td>$q+2$</td><td colspan="3">$2q+2$</td><td colspan="3"></td></tr>
<tr><td>3</td><td>1</td><td>line and 3·</td><td>$q+4$</td><td>$2q+3$</td><td>$3q+2$</td><td>$3q+1$</td><td>$2q+1$</td><td>$q+1$</td><td></td></tr>
<tr><td>4</td><td>1</td><td>line and 2·</td><td>$q+3$</td><td>$2q+2$</td><td>$3q+1$</td><td>$2q+1$</td><td>$q+1$</td><td colspan="2"></td></tr>
<tr><td>5</td><td>1</td><td>line and 1·</td><td>$q+2$</td><td colspan="2">$2q+1$</td><td>$q+1$</td><td colspan="3"></td></tr>
<tr><td>6</td><td>1</td><td>line</td><td colspan="3">$q+1$</td><td colspan="4"></td></tr>
<tr><td>2</td><td>0</td><td>9·</td><td>9</td><td>8</td><td>7</td><td>6</td><td>5</td><td>4</td><td>3</td></tr>
<tr><td>2</td><td>0</td><td>× + 5·</td><td>6</td><td>$q+6$</td><td>$q+5$</td><td>$q+4$</td><td>$q+3$</td><td>$q+2$</td><td>$q+1$</td></tr>
<tr><td>2</td><td>0</td><td>× ×</td><td>2</td><td>$q+2$</td><td>$2q+2$</td><td colspan="4"></td></tr>
<tr><td>2</td><td>0</td><td>triple point</td><td>1</td><td>$q+1$</td><td colspan="5"></td></tr>
<tr><td>3</td><td>0</td><td>7·</td><td>7</td><td>6</td><td>5</td><td>4</td><td>3</td><td>2</td><td>1</td></tr>
<tr><td>3</td><td>0</td><td>× + 3·</td><td>4</td><td>$q+4$</td><td>$q+3$</td><td>$q+2$</td><td>$q+1$</td><td colspan="2"></td></tr>
<tr><td>4</td><td>0</td><td>6·</td><td>6</td><td>5</td><td>4</td><td>3</td><td>2</td><td>1</td><td>0</td></tr>
<tr><td>4</td><td>0</td><td>× + 2·</td><td>3</td><td>$q+3$</td><td>$q+2$</td><td>$q+1$</td><td colspan="3"></td></tr>
<tr><td>5</td><td>0</td><td>5·</td><td>5</td><td>4</td><td>3</td><td>2</td><td>1</td><td>0</td><td></td></tr>
<tr><td>5</td><td>0</td><td>× + 1·</td><td>2</td><td>$q+2$</td><td>$q+1$</td><td colspan="4"></td></tr>
<tr><td>6</td><td>0</td><td>4·</td><td>4</td><td>3</td><td>2</td><td>1</td><td>0</td><td colspan="2"></td></tr>
<tr><td>6</td><td>0</td><td>×</td><td>1</td><td>$q+1$</td><td colspan="5"></td></tr>
<tr><td>7</td><td>0</td><td>3·</td><td>3</td><td>2</td><td>1</td><td>0</td><td colspan="3"></td></tr>
<tr><td>8</td><td>0</td><td>2·</td><td>2</td><td>1</td><td>0</td><td colspan="4"></td></tr>
<tr><td>9</td><td>0</td><td>1·</td><td>1</td><td>0</td><td colspan="5"></td></tr>
</table>

TABLE 3. Generalized Hamming Weights for Codes on the Del Pezzo Surfaces

r	$\mathcal{D}^9$ q^2+q+1	$\mathcal{D}^8$ q^2+2q+1	$\mathcal{D}^7$ q^2+3q+1	$\mathcal{D}^6$ q^2+4q+1	$\mathcal{D}^5$ q^2+5q+1	$\mathcal{D}^4$ q^2+6q+1	$\mathcal{D}^3$ q^2+7q+1
1	q^2-2q	q^2-2q	q^2-2q	q^2-2q+1	q^2	q^2+2q	q^2+4q+1
2	q^2-q+1	q^2-q+1	q^2-q	q^2	q^2+2q	q^2+4q	q^2+6q
3	q^2-q	q^2-q	q^2-1	q^2+q	q^2+3q	q^2+5q	q^2+7q
4	q^2-2	q^2-1	q^2	q^2+2q	q^2+4q	q^2+6q	q^2+7q+1
5	q^2-1	q^2	q^2+q-1	q^2+3q	q^2+5q	q^2+6q+1	
6	q^2	q^2+q	q^2+2q	q^2+4q	q^2+5q+1		
7	q^2+q-2	q^2+2q-1	q^2+3q	q^2+4q+1			
8	q^2+q-1	q^2+2q	q^2+3q+1				
9	q^2+q	q^2+2q+1					
10	q^2+q+1						

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